Exercise 8

Evaluate the line integral, where C is the given curve.

$$\int_C x^2\,dx + y^2\,dy, \quad C \text{ consists of the arc of the circle } x^2+y^2=4 \text{ from } (2,0) \text{ to } (0,2)$$
$$\text{followed by the line segment from } (0,2) \text{ to } (4,3)$$

Solution

Begin by writing the integral in terms of a dot product.

$$\int_C x^2\,dx + y^2\,dy = \int_C \langle x^2, y^2\rangle \cdot \langle dx, dy\rangle$$

Split up the line integral over the circular arc and the line segment.

$$\int_C x^2\,dx + y^2\,dy = \int_{\text{Arc}} \langle x^2, y^2\rangle \cdot \langle dx, dy\rangle + \int_{\text{Line}} \langle x^2, y^2\rangle \cdot \langle dx, dy\rangle$$

Parameterize the arc by setting $x = 2\cos t$ and $y = 2\sin t$ with $0 \le t \le \pi/2$. The equation of the line going through $(0,2)$ and $(4,3)$ is $y = (1/4)x + 2$; parameterize it by setting $x = t$, which then means $y = (1/4)t + 2$, with $0 \le t \le 4$. With these parameterizations in t, the line integral becomes

$$\begin{aligned}
\int_C x^2\,dx + y^2\,dy &= \int_0^{\pi/2} \langle [x(t)]^2, [y(t)]^2\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle dt \\
&\qquad + \int_0^4 \langle [x(t)]^2, [y(t)]^2\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle dt \\
&= \int_0^{\pi/2} \langle (2\cos t)^2, (2\sin t)^2\rangle \cdot \langle -2\sin t, 2\cos t\rangle\,dt \\
&\qquad + \int_0^4 \left\langle (t)^2, \left(\frac{1}{4}t + 2\right)^2\right\rangle \cdot \left\langle 1, \frac{1}{4}\right\rangle dt \\
&= \int_0^{\pi/2} \langle 4\cos^2 t, 4\sin^2 t\rangle \cdot \langle -2\sin t, 2\cos t\rangle\,dt \\
&\qquad + \int_0^4 \left\langle t^2, \frac{1}{16}t^2 + t + 4\right\rangle \cdot \left\langle 1, \frac{1}{4}\right\rangle dt \\
&= \int_0^{\pi/2} (-8\cos^2 t\sin t + 8\sin^2 t\cos t)\,dt + \int_0^4 \left[t^2 + \frac{1}{4}\left(\frac{1}{16}t^2 + t + 4\right)\right] dt \\
&= -8\int_0^{\pi/2} \cos^2 t\sin t\,dt + 8\int_0^{\pi/2} \sin^2 t\cos t\,dt + \int_0^4 \left(\frac{65}{64}t^2 + \frac{1}{4}t + 1\right) dt.
\end{aligned}$$

Make the following substitutions in the first and second integrals.

$$\begin{aligned}
u &= \cos t & v &= \sin t \\
du &= -\sin t\,dt & dv &= \cos t\,dt
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_C x^2\,dx + y^2\,dy &= -8\int_{\cos(0)}^{\cos(\pi/2)} u^2(-du) + 8\int_{\sin(0)}^{\sin(\pi/2)} v^2(dv) + \int_0^4 \left(\frac{65}{64}t^2 + \frac{1}{4}t + 1\right) dt \\
&= -8\int_1^0 u^2(-du) + 8\int_0^1 v^2(dv) + \int_0^4 \left(\frac{65}{64}t^2 + \frac{1}{4}t + 1\right) dt \\
&= \cancel{-8\int_0^1 u^2\,du} + \cancel{8\int_0^1 v^2\,dv} + \int_0^4 \left(\frac{65}{64}t^2 + \frac{1}{4}t + 1\right) dt \\
&= \left(\frac{65}{192}t^3 + \frac{1}{8}t^2 + t\right)\Bigg|_0^4 \\
&= \frac{65}{192}(4)^3 + \frac{1}{8}(4)^2 + (4) \\
&= \frac{83}{3}.
\end{aligned}$$